SEVEN THEOREMS IN THE PROBLEM OF PLATEAU

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The writer has recently completed the redaction of the manuscript of a paper presenting in complete form his results on the problem of Plateau for two contours; most of these results have been in my possession for some time past.[1] They are embodied in the form of seven theorems, which it is the purpose of this note to state. The complete paper will be published in the *Journal of Mathematics and Physics* of the Massachusetts I stitute of Technology.

The two contours Γ_1, Γ_2 are Jordan curves in euclidean space, of n dimensions for theorems I, V, of two dimensions for theorem IV, and of three dimensions for theorems II, III, VI, VII. Always, Γ_1 *and* Γ_2 *are supposed not to intersect one another.*

With Γ_1, Γ_2 are associated three positive numbers, finite or $+\infty$:

$$m(\Gamma_1), \quad m(\Gamma_2), \quad m(\Gamma_1, \Gamma_2).$$

Concretely, these are, respectively, the least areas that can be bounded by Γ_1, by Γ_2, by Γ_1 and Γ_2;[2] but for our analysis they are the lower bounds of certain functionals

$$A(g_1), A(g_2), A(g_1, g_2; q),$$

where g_1, g_2 are arbitrary parametric representations of Γ_1, Γ_2 and q is a parameter, $0 < q < 1$. Always, there is the inequality

$$m(\Gamma_1, \Gamma_2) \leqq m(\Gamma_1) + m(\Gamma_2),$$

or

$$e(\Gamma_1, \Gamma_2) = m(\Gamma_1) + m(\Gamma_2) - m(\Gamma_1, \Gamma_2) \geqq 0.$$

The functional of pairs of contours $e(\Gamma_1, \Gamma_2)$ is defined by the last formula for the case of finite $m(\Gamma_1, \Gamma_2)$. When $m(\Gamma_1, \Gamma_2) = +\infty$, we use the functional:

$$\bar{e}(\Gamma_1, \Gamma_2) = \limsup e(\Gamma_1', \Gamma_2') \geqq 0,$$

where Γ_1', Γ_2', contours with finite $m(\Gamma_1', \Gamma_2')$, (e.g., polygons), tend to Γ_1, Γ_2.

In theorems I, II, III, $m(\Gamma_1, \Gamma_2)$ is supposed finite, while in theorems V, VI, VII the contours are arbitrary Jordan curves, generally with $m(\Gamma_1, \Gamma_2) = +\infty$.

Minimal surface means one defined by the Weierstrass formulas:

$$x_i = R\ F_i(w),$$
$$\sum_{i=1}^{n} F_i'^2\ (w) = 0.$$

THEOREM I. *Let Γ_1, Γ_2 be two Jordan curves not intersecting one another; let $m(\Gamma_1, \Gamma_2)$ be finite, and suppose we have the strict inequality:*

$$m(\Gamma_1, \Gamma_2) < m(\Gamma_1) + m(\Gamma_2),$$

or

$$e(\Gamma_1, \Gamma_2) > 0.$$

Then there exists a doubly-connected minimal surface bounded by Γ_1, Γ_2. The area of this surface is $m(\Gamma_1, \Gamma_2)$.

THEOREM II. *Let Γ_1, Γ_2 be two Jordan curves not intersecting one another, and let $m(\Gamma_1, \Gamma_2)$ be finite.*

If the minimal surfaces M_1 and M_2, determined by Γ_1 and Γ_2 taken separately,[3] *have in common a point that is regular for both of them $\left(\sum_{i=1}^{n} |F_i'(w)|^2 > 0\right)$, then there exists a doubly-connected minimal surface bounded by Γ_1 and Γ_2.*

The area of this surface is less than the sum of the areas of M_1 and M_2.

THEOREM III. *Let Γ_1 and Γ_2, Jordan curves with finite $m(\Gamma_1, \Gamma_2)$, interlace. Then Γ_1, Γ_2 are the boundaries of a doubly-connected minimal surface.*

The writer's theory of the problem of Plateau includes the conformal mapping of plane regions as the special case $n = 2$. It is in this sense that the following theorem is to be understood.

THEOREM IV. *Let Γ_1, Γ_2 denote any two Jordan curves in the plane which enclose between them a region R. In the equations*

$$Z = g_1(z),\ Z = g_2(z),$$

where Z and z denote complex variables, let Z describe Γ_1, Γ_2, respectively, when z describes two concentric circles C_1, C_2 of radii $1, q$; $0 < q < 1$.

The range of values of the functional

$$A(g_1, g_2;\ q) = \frac{1}{4\pi} \sum_{\alpha\beta} \int_{C_\alpha} \int_{C_\beta} |\ g_\alpha(z) - g_\beta(\zeta)\ |^{\,2}\ P(z, \zeta;\ q)dz\ d\zeta,$$[4]

$\left(P(z, \zeta;\ q)\right.$ being a certain elliptic function with periods 2π, $2\sqrt{-1}\log\frac{1}{q}\Big)$, when all parametric representations g_1, g_2 of Γ_1, Γ_2 and all values of q are considered, will consist exactly of all positive real numbers $\geqq$ the inner area[5] *of the region R. This minimum value will be attained for a certain (essentially*

uniquely determined) parametric representation

$$Z = g_1^*(z), Z = g_2^*(z),$$

together with a unique value q^* of q.

Then the integral formula of Cauchy:

$$W = \frac{1}{2\pi i}\int_{C_1}\frac{g_1^*(z)\,dz}{z-w} + \frac{1}{2\pi i}\int_{C_2}\frac{g_2^*(z)\,dz}{z-w},$$ [4]

defines a conformal transformation $w \longrightarrow W$ of the circular ring between C_1 and C_2 into the region R between Γ_1 and Γ_2; this conformal transformation, furthermore, attaches continuously to the topological correspondence g_1^*, g_2^* between the boundaries.

In theorems V, VI, VII the restriction of finite $m(\Gamma_1, \Gamma_2)$ is removed from theorems I, II, III.

THEOREM V. *Any two Jordan curves Γ_1, Γ_2, not intersecting one another, for which*

$$\bar{e}(\Gamma_1, \Gamma_2) > 0,$$

are the boundaries of a doubly-connected minimal surface.

THEOREM VI. *If Γ_1, Γ_2 are any two Jordan curves not intersecting one another, and the minimal surfaces M_1 and M_2 determined by Γ_1 and Γ_2 separately*[3] *have a regular point in common, then there exists a doubly-connected minimal surface bounded by Γ_1, Γ_2.*

THEOREM VII. *Any two interlacing Jordan curves are the boundaries of a doubly-connected minimal surface.*

[1] "A General Formulation of the Problem of Plateau," presented to the American Mathematical Society, Oct. 26, 1929, abstract in *Bull. Am. Math. Soc.*, **36**, 50 (1930). "The Problem of Plateau for Two Contours," communicated to the same society, Sept. 27, 1930, abstract in the same publication, **36**, 797 (1930).

[2] The surfaces bounded by Γ_1 and Γ_2 separately are supposed to be simply-connected; those bounded by Γ_1, Γ_2 jointly, doubly-connected.

[3] The existence of the minimal surfaces M_1 and M_2 is assured by the writer's paper "Solution of the Problem of Plateau," *Trans. Amer. Math. Soc.*, **33**, 1, 263–321 (Jan., 1931), which gave the first general solution of the Plateau problem for a single contour.

[4] The sense of integration around C_1, C_2 is such that the circular ring between them is on the left.

[5] The upper bound of the area of a ring-shaped polygon whose boundaries P_1, P_2 encircle Γ_2 and are encircled by Γ_1; this is not always the same as the lower bound of the area of a ring-shaped polygon whose outer boundary P_1 encircles Γ_1 and whose inner boundary P_2 is encircled by Γ_2 (outer area).